

# STABILITY OF SPACELIKE HYPERSURFACES IN FOLIATED SPACETIMES

A. BARROS, A. BRASIL AND A. CAMINHA

**ABSTRACT.** Given a generalized  $\overline{M}^{n+1} = I \times_{\phi} F^n$  Robertson-Walker spacetime we will classify strongly stable spacelike hypersurfaces with constant mean curvature whose warping function verifies a certain convexity condition. More precisely, we will show that given  $x : M^n \rightarrow \overline{M}^{n+1}$  a closed spacelike hypersurfaces of  $\overline{M}^{n+1}$  with constant mean curvature  $H$  and the warping function  $\phi$  satisfying  $\phi'' \geq \max\{H\phi', 0\}$ , then  $M^n$  is either minimal or a spacelike slice  $M_{t_0} = \{t_0\} \times F$ , for some  $t_0 \in I$ .

## 1. INTRODUCTION

Spacelike hypersurfaces with constant mean curvature in Lorentz manifolds have been object of great interest in recent years, both from physical and mathematical points of view. In [1], the authors studied the uniqueness of spacelike hypersurfaces with CMC in generalized Robertson-Walker (GRW) spacetimes, namely, Lorentz warped products with 1-dimensional negative definite base and Riemannian fiber. They proved that in a GRW spacetime obeying the timelike convergence condition (i.e, the Ricci curvature is non-negative on timelike directions), every compact spacelike hypersurface with CMC must be umbilical. Recently, Alías and Montiel obtained, in [2], a more general condition on the warping function  $f$  that is sufficient in order to guarantee uniqueness. More precisely, they proved the following

**Theorem 1.1.** *Let  $f : I \rightarrow \mathbb{R}$  be a positive smooth function defined on an open interval, such that  $ff'' - (f')^2 \leq 0$ , that is, such that  $-\log f$  is convex. Then, the only compact spacelike hypersurfaces immersed into a generalized Robertson-Walker spacetime  $I \times_f F^n$  and having constant mean curvature are the slices  $\{t\} \times F$ , for a (necessarily compact) Riemannian manifold  $F$ .*

Stability questions concerning CMC, compact hypersurfaces in Riemannian space forms began with Barbosa and do Carmo in [4], and Barbosa, Do Carmo and Escobar in [5]. In the former paper, they introduced the notion of stability and proved that spheres are the only stable critical points for the area functional, for volume-preserving variations. In the setting of spacelike hypersurfaces in Lorentz manifolds, Barbosa and Olikar proved in [6] that CMC spacelike hypersurfaces are critical points of volume-preserving variations. Moreover, by computing the second variation formula they showed that CMC embedded spheres in the de Sitter space  $S_1^{n+1}$  maximize the area functional for such variations. In this paper, we give a characterization of *strongly stable*, CMC spacelike hypersurfaces in GRW spacetimes, the essential tool for the proof being a formula for the Laplacian of a new support function. More precisely, it is our purpose to show the following

**Theorem 1.2.** *Let  $\overline{M}^{n+1} = I \times_\phi F^n$  be a generalized Robertson-Walker spacetime, and  $x : M^n \rightarrow \overline{M}^{n+1}$  be a closed spacelike hypersurface of  $\overline{M}^{n+1}$ , having constant mean curvature  $H$ . If the warping function  $\phi$  satisfies  $\phi'' \geq \max\{H\phi', 0\}$  and  $M^n$  is strongly stable, then  $M^n$  is either minimal or a spacelike slice  $M_{t_0} = \{t_0\} \times F$ , for some  $t_0 \in I$ .*

## 2. STABLE SPACELIKE HYPERSURFACES

In what follows,  $\overline{M}^{n+1}$  denotes an orientable, time-oriented Lorentz manifold with Lorentz metric  $\overline{g} = \langle \cdot, \cdot \rangle$  and semi-Riemannian connection  $\overline{\nabla}$ . If  $x : M^n \rightarrow \overline{M}^{n+1}$  is a spacelike hypersurface of  $\overline{M}^{n+1}$ , then  $M^n$  is automatically orientable ([8], p. 189), and one can choose a globally defined unit normal vector field  $N$  on  $M^n$  having the same time-orientation of  $V$ , that is, such that

$$\langle V, N \rangle < 0$$

on  $M$ . One says that such an  $N$  *points to the future*.

A *variation* of  $x$  is a smooth map

$$X : M^n \times (-\epsilon, \epsilon) \rightarrow \overline{M}^{n+1}$$

satisfying the following conditions:

- (1) For  $t \in (-\epsilon, \epsilon)$ , the map  $X_t : M^n \rightarrow \overline{M}^{n+1}$  given by  $X_t(p) = X(t, p)$  is a spacelike immersion such that  $X_0 = x$ .
- (2)  $X_t|_{\partial M} = x|_{\partial M}$ , for all  $t \in (-\epsilon, \epsilon)$ .

The *variational field* associated to the variation  $X$  is the vector field  $\frac{\partial X}{\partial t}$ . Letting  $f = -\langle \frac{\partial X}{\partial t}, N \rangle$ , we get

$$\frac{\partial X}{\partial t} \Big|_M = fN + \left( \frac{\partial X}{\partial t} \right)^T,$$

where  $T$  stands for tangential components. The *balance of volume* of the variation  $X$  is the function  $\mathcal{V} : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$  given by

$$\mathcal{V}(t) = \int_{M \times [0, t]} X^*(d\overline{M}),$$

where  $d\overline{M}$  denotes the volume element of  $\overline{M}$ .

The *area functional*  $\mathcal{A} : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$  associated to the variation  $X$  is given by

$$\mathcal{A}(t) = \int_M dM_t,$$

where  $dM_t$  denotes the volume element of the metric induced in  $M$  by  $X_t$ . Note that  $dM_0 = dM$  and  $\mathcal{A}(0) = \mathcal{A}$ , the volume of  $M$ . The following lemma is classical:

**Lemma 2.1.** *Let  $\overline{M}^{n+1}$  be a time-oriented Lorentz manifold and  $x : M^n \rightarrow \overline{M}^{n+1}$  a spacelike closed hypersurface having mean curvature  $H$ . If  $X : M^n \times (-\epsilon, \epsilon) \rightarrow \overline{M}^{n+1}$  is a variation of  $x$ , then*

$$\frac{d\mathcal{V}}{dt} \Big|_{t=0} = \int_M f dM, \quad \frac{d\mathcal{A}}{dt} \Big|_{t=0} = \int_M nH f dM.$$

Set  $H_0 = \frac{1}{A} \int_M dM$  and  $\mathcal{J} : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$  given by

$$\mathcal{J}(t) = A(t) - nH_0\mathcal{V}(t).$$

$\mathcal{J}$  is called the *Jacobi functional* associated to the variation, and it is a well known result [5] that  $x$  has constant mean curvature  $H_0$  if and only if  $\mathcal{J}'(0) = 0$  for all variations  $X$  of  $x$ .

We wish to study here immersions  $x : M^n \rightarrow \overline{M}^{n+1}$  that maximize  $\mathcal{J}$  for all variations  $X$ . Since  $x$  must be a critical point of  $\mathcal{J}$ , it thus follows from the above discussion that  $x$  must have constant mean curvature. Therefore, in order to examine whether or not some critical immersion  $x$  is actually a maximum for  $\mathcal{J}$ , one certainly needs to study the second variation  $\mathcal{J}''(0)$ . We start with the following

**Proposition 2.2.** *Let  $x : M^n \rightarrow \overline{M}^{n+1}$  be a closed spacelike hypersurface of the time-oriented Lorentz manifold  $\overline{M}^{n+1}$ , and  $X : M^n \times (-\epsilon, \epsilon) \rightarrow \overline{M}^{n+1}$  be a variation of  $x$ . Then,*

$$(2.1) \quad n \frac{\partial H}{\partial t} = \Delta f - \{\overline{Ric}(N, N) + |A|^2\} f - n \left\langle \left( \frac{\partial X}{\partial t} \right)^T, \nabla H \right\rangle.$$

Although the above proposition is known to be true, we believe there is a lack, in the literature, of a clear proof of it in this degree of generality, so we present a simple proof here.

*Proof.* Let  $p \in M$  and  $\{e_k\}$  be a moving frame on a neighborhood  $U \subset M$  of  $p$ , geodesic at  $p$  and diagonalizing  $A$  at  $p$ , with  $Ae_k = \lambda_k e_k$  for  $1 \leq k \leq n$ . Extend  $N$  and the  $e'_k$ s to a neighborhood of  $p$  in  $\overline{M}$ , so that  $\langle N, e_k \rangle = 0$  and  $(\overline{\nabla}_N e_k)(p) = 0$ . Then

$$\begin{aligned} n \frac{\partial H}{\partial t} &= -\text{tr} \left( \frac{\partial A}{\partial t} \right) = -\sum_k \left\langle \frac{\partial A}{\partial t} e_k, e_k \right\rangle = -\sum_k \left\langle \left( \overline{\nabla}_{\frac{\partial X}{\partial t}} A \right) e_k, e_k \right\rangle \\ &= -\sum_k \left\{ \left\langle \overline{\nabla}_{\frac{\partial X}{\partial t}} A e_k, e_k \right\rangle - \left\langle A \overline{\nabla}_{\frac{\partial X}{\partial t}} e_k, e_k \right\rangle \right\} \\ &= \sum_k \left\langle \overline{\nabla}_{\frac{\partial X}{\partial t}} \overline{\nabla}_{e_k} N, e_k \right\rangle + \sum_k \left\langle A \overline{\nabla}_{e_k} \frac{\partial X}{\partial t}, e_k \right\rangle, \end{aligned}$$

where in the last equality we used the fact that  $[\frac{\partial X}{\partial t}, e_k] = 0$ . Letting

$$I = \sum_k \left\langle \overline{\nabla}_{\frac{\partial X}{\partial t}} \overline{\nabla}_{e_k} N, e_k \right\rangle \quad \text{and} \quad II = \sum_k \left\langle A \overline{\nabla}_{e_k} \frac{\partial X}{\partial t}, e_k \right\rangle,$$

we have

$$\begin{aligned} I &= \sum_k \left\{ \left\langle \overline{\nabla}_{\frac{\partial X}{\partial t}} \overline{\nabla}_{e_k} N - \overline{\nabla}_{e_k} \overline{\nabla}_{\frac{\partial X}{\partial t}} N + \overline{\nabla}_{[e_k, \frac{\partial X}{\partial t}]} N, e_k \right\rangle + \left\langle \overline{\nabla}_{e_k} \overline{\nabla}_{\frac{\partial X}{\partial t}} N, e_k \right\rangle \right\} \\ &= \sum_k \left\{ \left\langle \overline{R} \left( e_k, \frac{\partial X}{\partial t} \right) N, e_k \right\rangle + \left\langle \overline{\nabla}_{e_k} \overline{\nabla}_{\frac{\partial X}{\partial t}} N, e_k \right\rangle \right\} \\ &= -\overline{Ric} \left( \frac{\partial X}{\partial t}, N \right) + \sum_k \left\langle \overline{\nabla}_{e_k} \overline{\nabla}_{\frac{\partial X}{\partial t}} N, e_k \right\rangle. \end{aligned}$$

Since the frame  $\{e_k\}$  is geodesic at  $p$ , it follows that

$$\left\langle \overline{\nabla}_{\frac{\partial X}{\partial t}} N, \overline{\nabla}_{e_k} e_k \right\rangle = \left\langle \overline{\nabla}_{\frac{\partial X}{\partial t}} N, N \right\rangle \left\langle \overline{\nabla}_{e_k} e_k, N \right\rangle = 0$$

at  $p$ , and hence

$$\begin{aligned}
\langle \bar{\nabla}_{e_k} \bar{\nabla}_{\frac{\partial X}{\partial t}} N, e_k \rangle &= e_k \langle \bar{\nabla}_{\frac{\partial X}{\partial t}} N, e_k \rangle = -e_k \langle N, \bar{\nabla}_{\frac{\partial X}{\partial t}} e_k \rangle = -e_k \langle N, \bar{\nabla}_{e_k} \frac{\partial X}{\partial t} \rangle \\
&= -e_k e_k \langle N, \frac{\partial X}{\partial t} \rangle + e_k \langle \bar{\nabla}_{e_k} N, \frac{\partial X}{\partial t} \rangle \\
&= e_k e_k(f) + e_k \langle \bar{\nabla}_{e_k} N, \left( \frac{\partial X}{\partial t} \right)^T \rangle \\
&= e_k e_k(f) + \langle \bar{\nabla}_{e_k} \bar{\nabla}_{e_k} N, \left( \frac{\partial X}{\partial t} \right)^T \rangle - \langle A e_k, \bar{\nabla}_{e_k} \left( \frac{\partial X}{\partial t} \right)^T \rangle.
\end{aligned}$$

For  $II$ , we have

$$\begin{aligned}
II &= \sum_k \langle A e_k, \bar{\nabla}_{e_k} \frac{\partial X}{\partial t} \rangle = \sum_k \langle A e_k, \bar{\nabla}_{e_k} (f N + \left( \frac{\partial X}{\partial t} \right)^T) \rangle \\
&= \sum_k \langle A e_k, f \bar{\nabla}_{e_k} N \rangle + \sum_k \langle A e_k, \bar{\nabla}_{e_k} \left( \frac{\partial X}{\partial t} \right)^T \rangle \\
&= -f |A|^2 + \sum_k \langle A e_k, \bar{\nabla}_{e_k} \left( \frac{\partial X}{\partial t} \right)^T \rangle
\end{aligned}$$

Therefore,

$$(2.2) \quad n \frac{\partial H}{\partial t} = -\overline{Ric} \left( \frac{\partial X}{\partial t}, N \right) + \Delta f - f |A|^2 + \sum_k \langle \bar{\nabla}_{e_k} \bar{\nabla}_{e_k} N, \left( \frac{\partial X}{\partial t} \right)^T \rangle.$$

Now, letting

$$\frac{\partial X}{\partial t} = \sum_l^n \alpha_l e_l + f N$$

and  $A e_k = \sum_j h_{jk} e_j$ , one successively gets

$$\begin{aligned}
\overline{Ric} \left( \frac{\partial X}{\partial t}, N \right) &= \sum_l \alpha_l \overline{Ric}(N, e_l) + f \overline{Ric}(N, N) \\
&= \sum_{k,l} \alpha_l \langle \bar{R}(e_k, e_l) e_k, N \rangle + f \overline{Ric}(N, N)
\end{aligned}$$

and, since  $(\bar{\nabla}_N e_k)(p) = 0$ ,

$$\begin{aligned}
\langle \bar{R}(e_k, e_l) e_k, N \rangle_p &= \langle \bar{\nabla}_{e_l} \bar{\nabla}_{e_k} e_k - \bar{\nabla}_{e_k} \bar{\nabla}_{e_l} e_k, N \rangle_p \\
&= e_l \langle \bar{\nabla}_{e_k} e_k, N \rangle_p - \langle \bar{\nabla}_{e_k} e_k, \bar{\nabla}_{e_l} N \rangle_p - e_k \langle \bar{\nabla}_{e_l} e_k, N \rangle_p \\
&= -e_l \langle e_k, \bar{\nabla}_{e_k} N \rangle_p + e_k \langle e_k, \bar{\nabla}_{e_l} N \rangle_p \\
&= e_l (h_{kk}) - e_k (h_{kl}),
\end{aligned}$$

so that

$$(2.3) \quad \overline{Ric} \left( \frac{\partial X}{\partial t}, N \right)_p = \sum_{k,l} \alpha_l e_l (h_{kk}) - \sum_{k,l} \alpha_l e_k (h_{kl}) + f \overline{Ric}(N, N)_p.$$

Also,

$$\begin{aligned}
\alpha_l \langle \bar{\nabla}_{e_k} \bar{\nabla}_{e_k} N, e_l \rangle &= \alpha_l \langle \nabla_{e_k} \bar{\nabla}_{e_k} N, e_l \rangle = -\alpha_l \sum_j \langle \nabla_{e_k} h_{kj} e_j, e_l \rangle \\
&= -\alpha_l \sum_j \{ e_k \langle h_{kj} \rangle \delta_{lj} + h_{kj} \langle \nabla_{e_k} e_j, e_l \rangle \} \\
&= -\alpha_l e_k \langle h_{kl} \rangle,
\end{aligned}$$

and hence

$$(2.4) \quad \sum_k \langle \bar{\nabla}_{e_k} \bar{\nabla}_{e_k} N, \left( \frac{\partial X}{\partial t} \right)^T \rangle = - \sum_{k,l} \alpha_l e_k \langle h_{kl} \rangle.$$

Substituting (2.3) and (2.4) into (2.2), we finally arrive at

$$\begin{aligned}
n \frac{\partial H}{\partial t} &= - \sum_{k,l} \alpha_l e_l \langle h_{kk} \rangle - f \overline{Ric}(N, N)_p + \Delta f - f|A|^2 \\
&= - \left( \frac{\partial X}{\partial t} \right)^T (nH) - f \overline{Ric}(N, N)_p + \Delta f - f|A|^2.
\end{aligned}$$

□

**Proposition 2.3.** *Let  $\overline{M}^{n+1}$  be a Lorentz manifold and  $x : M^n \rightarrow \overline{M}^{n+1}$  be a closed spacelike hypersurface having constant mean curvature  $H$ . If  $X : M^n \times (-\epsilon, \epsilon) \rightarrow \overline{M}^{n+1}$  is a variation of  $x$ , then*

$$(2.5) \quad \mathcal{J}''(0)(f) = \int_M f \{ \Delta f - (\overline{Ric}(N, N) + |A|^2) f \} dM.$$

*Proof.* In the notations of the above discussion, set  $f = f(0)$  and note that  $H(0) = H$ . It follows from lemma 2.1 that

$$\mathcal{J}'(t) = \int_M n \{ H(t) - H \} f(t) dM_t.$$

Therefore, differentiating with respect to  $t$  once more

$$\begin{aligned}
\mathcal{J}''(0) &= \int_M n H'(0) f(0) dM_0 + \int_M n \{ H(0) - H \} \frac{d}{dt} f(t) dM_t \Big|_{t=0} \\
&= \int_M n H'(0) f dM.
\end{aligned}$$

Taking into account that  $H$  is constant, relation (2.1) finally gives formula 2.5 □

It follows from the previous result that  $\mathcal{J}''(0) = \mathcal{J}''(0)(f)$  depends only on  $f \in C^\infty(M)$ , for which there exists a variation  $X$  of  $M^n$  such that  $(\frac{\partial X}{\partial t})^\perp = fN$ . Therefore, the following definition makes sense:

**Definition 2.4.** Let  $\overline{M}^{n+1}$  be a Lorentz manifold and  $x : M^n \rightarrow \overline{M}^{n+1}$  be a closed spacelike hypersurface having constant mean curvature  $H$ . We say that  $x$  is strongly stable if, for every function  $f \in C^\infty(M)$  for which there exists a variation  $X$  of  $M^n$  such that  $(\frac{\partial X}{\partial t})^\perp = fN$ , one has  $\mathcal{J}''(0)(f) \leq 0$ .

## 3. CONFORMAL VECTOR FIELDS

As in the previous section, let  $\overline{M}^{n+1}$  be a Lorentz manifold. A vector field  $V$  on  $\overline{M}^{n+1}$  is said to be *conformal* if

$$(3.1) \quad \mathcal{L}_V \langle \cdot, \cdot \rangle = 2\psi \langle \cdot, \cdot \rangle$$

for some function  $\psi \in C^\infty(\overline{M})$ , where  $\mathcal{L}$  stands for the Lie derivative of the Lorentz metric of  $\overline{M}$ . The function  $\psi$  is called the *conformal factor* of  $V$ .

Since  $\mathcal{L}_V(X) = [V, X]$  for all  $X \in \mathcal{X}(\overline{M})$ , it follows from the tensorial character of  $\mathcal{L}_V$  that  $V \in \mathcal{X}(\overline{M})$  is conformal if and only if

$$(3.2) \quad \langle \overline{\nabla}_X V, Y \rangle + \langle X, \overline{\nabla}_Y V \rangle = 2\psi \langle X, Y \rangle,$$

for all  $X, Y \in \mathcal{X}(\overline{M})$ . In particular,  $V$  is a Killing vector field relatively to  $\overline{g}$  if and only if  $\psi \equiv 0$ .

Any Lorentz manifold  $\overline{M}^{n+1}$ , possessing a globally defined, timelike conformal vector field is said to be a *conformally stationary spacetime*.

**Proposition 3.1.** *Let  $\overline{M}^{n+1}$  be a conformally stationary Lorentz manifold, with conformal vector field  $V$  having conformal factor  $\psi : \overline{M}^{n+1} \rightarrow \mathbb{R}$ . Let also  $x : M^n \rightarrow \overline{M}^{n+1}$  be a spacelike hypersurface of  $\overline{M}^{n+1}$ , and  $N$  a future-pointing, unit normal vector field globally defined on  $M^n$ . If  $f = \langle V, N \rangle$ , then*

$$(3.3) \quad \Delta f = n \langle V, \nabla H \rangle + f \{ \overline{\text{Ric}}(N, N) + |A|^2 \} + n \{ H\psi - N(\psi) \},$$

where  $\overline{\text{Ric}}$  denotes the Ricci tensor of  $\overline{M}$ ,  $A$  is the second fundamental form of  $x$  with respect to  $N$ ,  $H = -\frac{1}{n} \text{tr}(A)$  is the mean curvature of  $x$  and  $\nabla H$  denotes the gradient of  $H$  in the metric of  $M$ .

*Proof.* Fix  $p \in M$  and let  $\{e_k\}$  be an orthonormal moving frame on  $M$ , geodesic at  $p$ . Extend the  $e_k$  to a neighborhood of  $p$  in  $\overline{M}$ , so that  $(\overline{\nabla}_N e_k)(p) = 0$ , and let

$$V = \sum_l^n \alpha_l e_l - fN.$$

Then

$$\begin{aligned} f = \langle N, V \rangle \Rightarrow e_k(f) &= \langle \overline{\nabla}_{e_k} N, V \rangle + \langle N, \overline{\nabla}_{e_k} V \rangle \\ &= -\langle Ae_k, V \rangle + \langle N, \overline{\nabla}_{e_k} V \rangle, \end{aligned}$$

so that

$$\begin{aligned} \Delta f &= \sum_k e_k(e_k(f)) = -\sum_k e_k \langle Ae_k, V \rangle + \sum_k e_k \langle N, \overline{\nabla}_{e_k} V \rangle \\ (3.4) \quad &= -\sum_k \langle \overline{\nabla}_{e_k} Ae_k, V \rangle - 2 \sum_k \langle Ae_k, \overline{\nabla}_{e_k} V \rangle + \sum_k \langle N, \overline{\nabla}_{e_k} \overline{\nabla}_{e_k} V \rangle. \end{aligned}$$

Now, differentiating  $Ae_k = \sum_l h_{kl}e_l$  with respect to  $e_k$ , one gets at  $p$

$$\begin{aligned}
 \sum_k \langle \bar{\nabla}_{e_k} Ae_k, V \rangle &= \sum_{k,l} e_k(h_{kl}) \langle e_l, V \rangle + \sum_{k,l} h_{kl} \langle \bar{\nabla}_{e_k} e_l, V \rangle \\
 &= \sum_{k,l} \alpha_l e_k(h_{kl}) - \sum_{k,l} h_{kl} \langle \bar{\nabla}_{e_k} e_l, N \rangle \langle V, N \rangle \\
 &= \sum_{k,l} \alpha_l e_k(h_{kl}) - \sum_{k,l} h_{kl}^2 f \\
 (3.5) \qquad \qquad \qquad &= \sum_{k,l} \alpha_l e_k(h_{kl}) - f|A|^2.
 \end{aligned}$$

Asking further that  $Ae_k = \lambda_k e_k$  at  $p$  (which is always possible), we have at  $p$

$$(3.6) \qquad \sum_k \langle Ae_k, \bar{\nabla}_{e_k} V \rangle = \sum_k \lambda_k \langle e_k, \bar{\nabla}_{e_k} V \rangle = \sum_k \lambda_k \psi = -nH\psi.$$

In order to compute the last summand of (3.4), note that the conformality of  $V$  gives

$$\langle \bar{\nabla}_N V, e_k \rangle + \langle N, \bar{\nabla}_{e_k} V \rangle = 0$$

for all  $k$ . Hence, differentiating the above relation in the direction of  $e_k$ , we get

$$\langle \bar{\nabla}_{e_k} \bar{\nabla}_N V, e_k \rangle + \langle \bar{\nabla}_N V, \bar{\nabla}_{e_k} e_k \rangle + \langle \bar{\nabla}_{e_k} N, \bar{\nabla}_{e_k} V \rangle + \langle N, \bar{\nabla}_{e_k} \bar{\nabla}_{e_k} V \rangle = 0.$$

However, at  $p$  one has

$$\begin{aligned}
 \langle \bar{\nabla}_N V, \bar{\nabla}_{e_k} e_k \rangle &= -\langle \bar{\nabla}_N V, \langle \bar{\nabla}_{e_k} e_k, N \rangle N \rangle = -\langle \bar{\nabla}_N V, \lambda_k N \rangle \\
 &= -\lambda_k \psi \langle N, N \rangle = \lambda_k \psi
 \end{aligned}$$

and

$$\langle \bar{\nabla}_{e_k} N, \bar{\nabla}_{e_k} V \rangle = -\lambda_k \langle e_k, \bar{\nabla}_{e_k} V \rangle = -\lambda_k \psi,$$

so that

$$(3.7) \qquad \langle \bar{\nabla}_{e_k} \bar{\nabla}_N V, e_k \rangle + \langle N, \bar{\nabla}_{e_k} \bar{\nabla}_{e_k} V \rangle = 0$$

at  $p$ . On the other hand, since

$$[N, e_k](p) = (\bar{\nabla}_N e_k)(p) - (\bar{\nabla}_{e_k} N)(p) = \lambda_k e_k(p),$$

it follows from (3.7) that

$$\begin{aligned}
 \langle \bar{R}(N, e_k)V, e_k \rangle_p &= \langle \bar{\nabla}_{e_k} \bar{\nabla}_N V - \bar{\nabla}_N \bar{\nabla}_{e_k} V + \bar{\nabla}_{[N, e_k]} V, e_k \rangle_p \\
 &= -\langle N, \bar{\nabla}_{e_k} \bar{\nabla}_{e_k} V \rangle_p - N \langle \bar{\nabla}_{e_k} V, e_k \rangle_p + \langle \bar{\nabla}_{\lambda_k e_k} V, e_k \rangle_p \\
 &= -\langle N, \bar{\nabla}_{e_k} \bar{\nabla}_{e_k} V \rangle_p - N(\psi) + \lambda_k \psi,
 \end{aligned}$$

and hence

$$(3.8) \qquad \sum_k \langle N, \bar{\nabla}_{e_k} \bar{\nabla}_{e_k} V \rangle_p = -nN(\psi) - nH\psi - \overline{Ric}(N, V)_p$$

Finally,

$$\begin{aligned}
 \overline{Ric}(N, V) &= \sum_l \alpha_l \overline{Ric}(N, e_l) - f \overline{Ric}(N, N) \\
 &= \sum_{k,l} \alpha_l \langle \bar{R}(e_k, e_l)e_k, N \rangle - f \overline{Ric}(N, N),
 \end{aligned}$$

and

$$\begin{aligned}
\langle \overline{R}(e_k, e_l)e_k, N \rangle_p &= \langle \overline{\nabla}_{e_l} \overline{\nabla}_{e_k} e_k - \overline{\nabla}_{e_k} \overline{\nabla}_{e_l} e_k, N \rangle_p \\
&= e_l \langle \overline{\nabla}_{e_k} e_k, N \rangle_p - \langle \overline{\nabla}_{e_k} e_k, \overline{\nabla}_{e_l} N \rangle_p - e_k \langle \overline{\nabla}_{e_l} e_k, N \rangle_p \\
&\quad + \langle \overline{\nabla}_{e_l} e_k, \overline{\nabla}_{e_k} N \rangle_p \\
&= -e_l \langle e_k, \overline{\nabla}_{e_k} N \rangle_p + e_k \langle e_k, \overline{\nabla}_{e_l} N \rangle_p \\
&= e_l(h_{kk}) - e_k(h_{kl}),
\end{aligned}$$

so that

$$\overline{Ric}(N, V)_p = \sum_{k,l} \alpha_l e_l(h_{kk}) - \sum_{k,l} \alpha_l e_k(h_{kl}) - f \overline{Ric}(N, N)_p,$$

and it follows from (3.8) that

$$\begin{aligned}
\sum_k \langle N, \overline{\nabla}_{e_k} \overline{\nabla}_{e_k} V \rangle_p &= -nN(\psi) - nH\psi + V^T(nH) \\
(3.9) \qquad \qquad \qquad &+ \sum_{k,l} \alpha_l e_k(h_{kl}) + f \overline{Ric}(N, N).
\end{aligned}$$

Substituting (3.5), (3.6) and (3.9) into (3.4), one gets the desired formula (3.3).  $\square$

#### 4. APPLICATIONS

A particular class of conformally stationary spacetimes is that of *generalized Robertson-Walker* spacetimes [1], namely, warped products  $\overline{M}^{n+1} = I \times_\phi F^n$ , where  $I \subseteq \mathbb{R}$  is an interval with the metric  $-dt^2$ ,  $F^n$  is an  $n$ -dimensional Riemannian manifold and  $\phi : I \rightarrow \mathbb{R}$  is positive and smooth. For such a space, let  $\pi_I : \overline{M}^{n+1} \rightarrow I$  denote the canonical projection onto the  $I$ -factor. Then the vector field

$$V = (\phi \circ \pi_I) \frac{\partial}{\partial t}$$

is conformal, timelike and closed (in the sense that its dual 1-form is closed), with conformal factor  $\psi = \phi'$ , where the prime denotes differentiation with respect to  $t$ . Moreover, according to [7], for  $t_0 \in I$ , orienting the (spacelike) leaf  $M_{t_0}^n = \{t_0\} \times F^n$  by using the future-pointing unit normal vector field  $N$ , it follows that  $M_{t_0}$  has constant mean curvature

$$H = \frac{\phi'(t_0)}{\phi(t_0)}.$$

If  $\overline{M}^{n+1} = I \times_\phi F^n$  is a generalized Robertson-Walker spacetime and  $x : M^n \rightarrow \overline{M}^{n+1}$  is a complete spacelike hypersurface of  $\overline{M}^{n+1}$ , such that  $\phi \circ \pi_I$  is limited on  $M$ , then  $\pi_F|_M : M^n \rightarrow F^n$  is necessarily a covering map ([1]). In particular, if  $M^n$  is closed, then  $F^n$  is automatically closed.

One has the following corollary of proposition 3.1:

**Corollary 4.1.** *Let  $\overline{M}^{n+1} = I \times_\phi F^n$  be a generalized Robertson-Walker spacetime, and  $x : M^n \rightarrow \overline{M}^{n+1}$  a spacelike hypersurface of  $\overline{M}^{n+1}$ , having constant mean curvature  $H$ . Let also  $N$  be a future-pointing unit normal vector field globally defined on  $M^n$ . If  $V = (\phi \circ \pi_I) \frac{\partial}{\partial t}$  and  $f = \langle V, N \rangle$ , then*

$$(4.1) \qquad \Delta f = \{ \overline{Ric}(N, N) + |A|^2 \} f + n \left\{ H\phi' + \phi'' \langle N, \frac{\partial}{\partial t} \rangle \right\}.$$

where  $\overline{Ric}$  denotes the Ricci tensor of  $\overline{M}$ ,  $A$  is the second fundamental form of  $x$  with respect to  $N$ , and  $H = -\frac{1}{n}\text{tr}(A)$  is the mean curvature of  $x$ .

*Proof.* First of all,  $f = \langle V, N \rangle = \phi \langle N, \frac{\partial}{\partial t} \rangle$ , and it thus follows from (3.3) that

$$\Delta f = \{ \overline{Ric}(N, N) + |A|^2 \} f + n \{ H\phi' - N(\phi') \}.$$

However,

$$\overline{\nabla}\phi' = -\langle \overline{\nabla}\phi', \frac{\partial}{\partial t} \rangle \frac{\partial}{\partial t} = -\phi'' \frac{\partial}{\partial t},$$

so that

$$N(\phi') = \langle N, \overline{\nabla}\phi' \rangle = -\phi'' \langle N, \frac{\partial}{\partial t} \rangle$$

□

We can now state and prove our main result:

**Theorem 4.2.** *Let  $\overline{M}^{n+1} = I \times_{\phi} F^n$  be a generalized Robertson-Walker spacetime, and  $x : M^n \rightarrow \overline{M}^{n+1}$  be a closed spacelike hypersurface of  $\overline{M}^{n+1}$ , having constant mean curvature  $H$ . If the warping function  $\phi$  satisfies  $\phi'' \geq \max\{H\phi', 0\}$  and  $M^n$  is strongly stable, then  $M^n$  is either minimal or a spacelike slice  $M_{t_0} = \{t_0\} \times F$ , for some  $t_0 \in I$ .*

*Proof.* Since  $M^n$  is strongly stable, we have

$$0 \geq \mathcal{J}''(0)(g) = \int_M g \{ \Delta g - (\overline{Ric}(N, N) + |A|^2) g \} dM$$

for all  $g \in C^\infty(M)$  for which  $gN$  is the normal component of the variational field of some variation of  $M^n$ . In particular, if  $f = \langle V, N \rangle = \phi \langle N, \frac{\partial}{\partial t} \rangle$ , where  $V = (\phi \circ \pi_I) \frac{\partial}{\partial t}$ , and  $g = -f = -\langle V, N \rangle$ , then

$$\Delta g = \{ \overline{Ric}(N, N) + |A|^2 \} g - n \left\{ H\phi' + \phi'' \langle N, \frac{\partial}{\partial t} \rangle \right\}.$$

Therefore,  $M^n$  stable implies

$$0 \geq \int_M \phi \langle N, \frac{\partial}{\partial t} \rangle \left\{ H\phi' + \phi'' \langle N, \frac{\partial}{\partial t} \rangle \right\} dM$$

Letting  $\theta$  be the hyperbolic angle between  $N$  and  $\frac{\partial}{\partial t}$ , it follows from the reversed Cauchy-Schwarz inequality that  $\cosh \theta = -\langle N, \frac{\partial}{\partial t} \rangle$ , with  $\cosh \theta \equiv 1$  if and only if  $N$  and  $\frac{\partial}{\partial t}$  are collinear at every point, that is, if and only if  $M^n$  is a spacelike leaf  $M_{t_0}$  for some  $t_0 \in I$ . Hence,

$$0 \geq \int_M \phi \cosh \theta \{ -H\phi' + \phi'' \cosh \theta \} dM.$$

Now, notice that  $-H\phi' + \phi'' \cosh \theta \geq -\phi'' + \phi'' \cosh \theta$ , which gives

$$\phi \cosh \theta (-H\phi' + \phi'' \cosh \theta) \geq \phi \phi'' \cosh \theta (\cosh \theta - 1).$$

Therefore,

$$0 \geq \int_M \phi \cosh \theta (-H\phi' + \phi'' \cosh \theta) dM \geq \int_M \phi \phi'' \cosh \theta (\cosh \theta - 1) dM \geq 0,$$

and hence

$$\phi''(\cosh \theta - 1) = 0 \quad \text{and} \quad \phi'' = H\phi'$$

on  $M$ . If, for some  $p \in M$ , one has  $\phi''(p) = 0$ , then  $\phi'H = 0$  at  $p$ . If  $H \neq 0$ , then  $\phi'(p) = 0$ . But if this is the case, then proposition 7.35 of [8] gives that

$$\bar{\nabla}_V \frac{\partial}{\partial t} = \frac{\phi'}{\phi} V = 0$$

at  $p$  for any  $V$ , and  $M$  is totally geodesic at  $p$ . In particular,  $H = 0$ , a contradiction. Therefore, either  $\phi''(p) = 0$  for some  $p \in M$ , and  $M$  is minimal, or  $\phi'' \neq 0$  on all of  $M$ , whence  $\cosh \theta = 1$  always, and  $M$  is an umbilical leaf such that  $\phi'' = H\phi'$ .  $\square$

*Remark 4.3.* Note that  $\frac{\phi''}{\phi'} = H = \frac{\phi'}{\phi}$ , i.e.,  $\phi''\phi - (\phi')^2 = 0$ , which is a limit case of Alías and Montiel's timelike convergent condition.

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